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PASSAGE THROUGH A SEPARATRIX IN A RESONANCE PROBLEM WITH A SLOWLY-VARYING PARAMETER

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In attempts to give an evolutionary explanation for the resonances in the system of Saturn's satellites one model problem has been examined [1, 2] on evolution in a Hamiltonian system with one degree of freedom and with a Hamiltonian depending explicitly on "slow time" δt. During the evolution a point on the phase plane intersects a separatrix of the unperturbed problem and it becomes impossible to give a deterministic description of the motion as $\delta \to 0$. In the paper we discuss various concepts of probability and obtain formulas for an effective computation of this probability. We propose and substantiate a scheme for analyzing the problem by the averaging method allowing us to examine the point's motion as it passes through the separatrix. Analogously we examine the problem of the evolution of a pendulum's motion with a slowly-varying frequency and a small external moment. Certain aspects of the problem of studying the behavior of a dynamic system as it passes through a separatrix of the unperturbed problem were examined in [3–5].

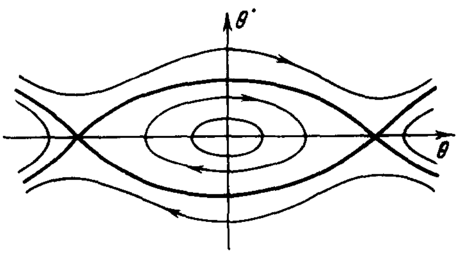

Fig. 1

We clarify the main problem of the present paper by example of the perturbed motion of a mathematical pendulum, which is described by the equation

$$\theta^{\cdot\cdot} + \omega^2 \sin\theta = -L \qquad (0.1)$$

We assume that the external moment L is independent of time and satisfies the conditions $L > 0$, $L = O(\delta)$, while the frequency ω grows slowly with time : $\omega^{\cdot} = O(\delta)$, where δ is a sufficiently small quantity. Regions of direct rotation $(\theta^{\cdot} > 0)$, of reverse rotation $(\theta^{\cdot} < 0)$ and of oscillatory motion, separated by a separatrix, occur on the phase plane of the unperturbed problem $(L = \omega^{\cdot} = 0)$ (Fig. 1). Two outcomes of evolution are possible, in general, for problem (0.1) of a pendulum initially moving in the direct rotation after a time of the order of $1/\delta$ it can shift into the reverse rotation or it can pass into the oscillatory mode. It can be shown that as $\delta \to 0$ the type of the final motion can be changed, for example, by a variation of the initial data by a quantity of the order of δ. This is connected with the effects of the pendulum's motion when passing a neighborhood of an osculating separatrix. A deterministic description of such a problem as $\delta \to 0$ is meaningless. We can speak only of the probability of this outcome or the other. In such problems there arise questions on the correct definition of probability and on the obtaining of an effective method of computing it, on the modification and substantiation of the applicability of the averaging method (with due regard to the probability arising here) for describing the solutions intersecting the separatrix. The thing is that the averaging method for describing evolutions on times of the order of $1/\delta$ has been justified and is applied usually only under the condition that the moving point is at a distance from the separatrix of the unperturbed problem. Whereas one of the peculiarities of the problems being considered here is the special interest in motion close to a separatrix.

1. On a model problem. Attempts to give an evolutionary explanantion of the resonance relations observable in the system of Saturn's satellites lead to the examination of a model problem [1, 2] defined by the Hamiltonian

$$F = 4\Gamma^2 - 2\lambda\Gamma + \mu\sqrt{2\Gamma}\cos\varphi \tag{1.1}$$

Here $\Gamma > 0$ and φ are adjoint canonic variables

$$\Gamma^\cdot = \partial F / \partial\varphi, \quad \varphi^\cdot = -\partial F / \partial\Gamma \tag{1.2}$$

$\mu > 0$ is a constant parameter, λ is a time function whose rate of change $\lambda^\cdot = \delta$ is assumed constant, positive and small: $0 < \delta \ll 1$. Together with Γ and φ it is convenient to use the canonic variables x and y in which the Hamiltonian F has the form

$$F = F(x, y, \lambda) = (x^2 + y^2)^2 - \lambda(x^2 + y^2) + \mu x \tag{1.3}$$

$$(x = \sqrt{2\Gamma}\cos\varphi, \; y = \sqrt{2\Gamma}\sin\varphi)$$

If we choose x and y as rectangular coordinates on a phase plane, then $e = \sqrt{2\Gamma}$ and φ are the polar coordinates.

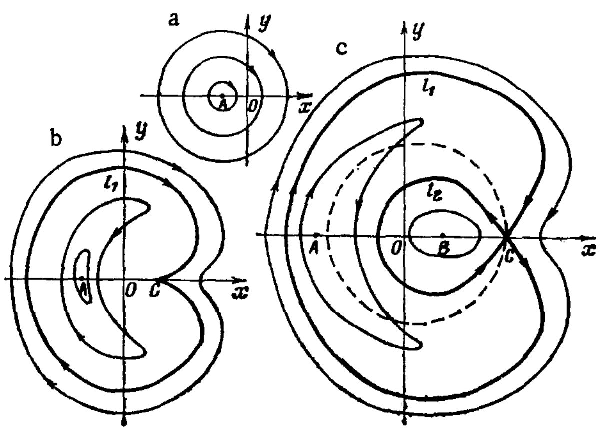

Fig. 2

2. Phase trajectories of the unperturbed problem. Since $\delta \ll 1$, as a preliminary we consider the problem with a fixed λ [1]. The behavior of the trajectories in the xy-plane as a function of the value of λ is shown in Fig. 2, a, b, c. For

$\lambda < \lambda_* = 1{,}5\ \mu^{2/3}$ (λ_* is the singular "critical" value of λ) a singular point A of center type exists in the xy-plane (Fig. 2, a) and all trajectories surround this point. For $\lambda > \lambda_*$ (Fig. 2, c) the xy-plane is divided into three regions by separatrices l_1 and l_2 : G_1, G_{12}, G_2. A singular point A of center type is contained inside the region G_{12}, while a singular point B of center type is contained within region G_2. Both separatrices start and end at a singular point C of saddle type. Figure 2, b shows the intermediate case of $\lambda = \lambda_*$.

Point C has the coordinates $(x_C, 0)$, where $x_C = x_C(\lambda)$ is the largest root of the equilibrium equation

$$\partial F(x, 0, \lambda) / \partial x = 4x^3 - 2\lambda x + \mu = 0 \tag{2.1}$$

When $\lambda \geqslant \lambda_*$ we introduce $F_C = F_C(\lambda) = F(x_C, 0, \lambda)$, the value of Hamiltonian F at point C, and the function $H(x, y, \lambda) = F(x, y, \lambda) - F_C(\lambda)$. Region G_{12} is given by the relation $H < 0$ and regions G_1 and G_2 by the relation $H > 0$. On the separatrices $H = 0$.

3. Qualitative description of the motion. We return to the problem with $\lambda^{\cdot} = \delta > 0$. When $\lambda \geqslant \lambda_*$, using (2.1) we once again introduce C, the singular point of the Hamiltonian, and in accordance with the equation $H = 0$ we introduce the curves l_1 and l_2 for which we retain the name separatrices. The separatrices divide the phase plane into regions G_1, G_{12}, G_2. Let G be the region included inside l_1. At times of the order of $1/\delta$ (the variation of λ is of the order of unity) there take place both a significant evolution of the motion as well as a deformation of the subdivision regions. As a result this point can intersect a separatrix and, having left one region, pass by its own evolution into another. It is precisely such a capture process we shall consider.

Let us state a number of assertions relative to the behavior of the solutions of the problem being analyzed, which have an asymptotic sense, i.e. are fulfilled for sufficiently small δ. We do not present the proofs of these assertions here because of their awkwardness; the text following the assertion is for explaining its meaning and the idea of the proof. By $(x(t), y(t))$ we denote a point in the phase plane, moving in accordance with (1.2). Without loss of generality we can assume that $\lambda = \lambda_*$ at the initial instant $t = 0$. Here the initial point $(x(0), y(0))$ can be found either inside l_1 (then $H_0 = H(x(0), y(0), \lambda_*) < 0$) or outside l_1 (then $H_0 > 0$).

Assertion 1. For any $k > \lambda_*$ there exist positive constants δ_0, k_1 such that if $\delta < \delta_0$ and $H_0 < -k_1\delta^{6/5}$, then $(x(t), y(t)) \in G_{12}$ at least as long as $\lambda < k$.

Thus, points lying inside l_1 when $\lambda = \lambda_*$ (Fig. 2, b), excepting, possibly, the points from a narrow belt adjacent to l_1 delineated by the condition $-k_1\delta^{6/5} \leqslant H_0 < 0$, remain in G_{12} at least for times of the order of $1/\delta$.

Note. The assertion that points occurring inside l_1 when $\lambda = \lambda_*$ remain in G_{12} was made in [1] wherein this case was called an "automatic capture". However, this assertion was not argued in any way in [1].

Assertion 2. If $H_0 > 0$, an instant t_1 exists at which the point hits upon l_1; for $t > t_1$ the point is found in region G. For sufficiently small δ the instant t_1 can be estimated as $t_1 < k_2/\delta$, where $k_2 > 0$ is a constant depending upon the initial conditions. If, in addition, $H_0 > k_3\delta$, where $k_3 > 0$ is a constant, then for $t > 0$ the point once for the last time hits onto the ray Cx before passing on into G.

To prove this we differentiate H relative to the equations of motion. We obtain

$$H^{\cdot} = \delta\, \partial H / \partial\lambda = \delta\,(\partial F / \partial\lambda - dF_C / d\lambda)$$

$$\frac{dF_C}{d\lambda} = \left(\frac{\partial F}{\partial\lambda}\right)_C + \left(\frac{\partial F}{\partial x}\right)_C \frac{dx_C}{d\lambda} = \left(\frac{\partial F}{\partial\lambda}\right)_C$$

$$H^{\cdot} = \delta\,[\partial F / \partial\lambda - (\partial F / \partial\lambda)_C\,] = -\,\delta\,(e^2 - e_C^2)$$

($(\partial F / \partial x)_C = 0$ since C is a singular point of F).

Lemma. The circle $e = e_C$ (with the exception of point C) lies in region G_{12} (this circle is shown in dotted line in Fig.2, c).

Corollary. $e > e_C$ is fulfilled in G_1 and on l_1; $e < e_C$ is fulfilled in G_2 and on l_2.

Therefore, H becomes equal to zero in G_1 along the motion $H^{\cdot} < 0$ and for some $t = t_1 < k_2 / \delta$ (see Sect. 7), i.e. the point goes onto l_1. The point is in G for $t > t_1$. Since the quantity $\varphi^{\cdot}$ does not vanish in G_1, the point approaches l_1 along a spiral. When $H_0 > k_3\delta$ the point is automatically able to make a complete turn in G_1. The value of H at the intersection of ray Cx by the last turn before going into G has been determined for such points.

Theorem 1. Suppose that for some $\lambda = \Lambda > \lambda_*$ the point is located on ray Cx. Then for sufficiently small δ ($\delta < \delta_0$, $\delta_0 > 0$ depends on Λ) we can find $\alpha_1'' > \alpha_1' > \alpha_2'' > \alpha_2' > \alpha_3'' > 0$ such that

1) If at this instant $H \in d = (\alpha_1'', \infty)$, the point makes a complete turn in G_1 and goes onto Cx.

2) If $H \in d_1 = (0, \alpha_1)$, intersects l_1, not going onto Cx any more; here, (2a) if $H \in d_2 = (\alpha_2'', \alpha_1')$, the point, not having reached the semicircle $e = e_C$, $y < 0$ goes onto l_2 and is subsequently found in G_2; (2b) if $H \in d_{12} = (\alpha_3'', \alpha_2')$, the point goes onto the semicircle $e = e_C$, $y < 0$, and for a sufficiently small $\delta_0 = \delta_0(\Lambda, k)$ remains in its subsequent motion in G_{12} at least as long as $\lambda < k$ for any $k > \Lambda$.

3) When starting from any of the intervals d, d_1, d_2, d_{12} the point goes onto the corresponding line (ray Cx, l_1, l_2 or $\{e = e_C;\ y < 0\}$ for $\lambda < \Lambda + k_4\delta\,|\ln\delta\,|$, $k_4 > 0$ is a constant).

4) The estimates

$$\alpha_1'' - \alpha_1' = \alpha_2'' - \alpha_2' = k_5\delta^{3/2}, \quad \alpha_3'' = k_6\delta^2 \tag{3.1}$$

$$(\alpha_1' + \alpha_1'') / 2 = \delta I_1(\Lambda), \quad (\alpha_2' + \alpha_2'') / 2 = \delta I_{12}(\Lambda)$$

$$I_1(\lambda) = -\oint_{l_1}\left(\frac{\partial H}{\partial\lambda}\right)dt, \quad I_2(\lambda) = \oint_{l_2}\left(\frac{\partial H}{\partial\lambda}\right)dt \tag{3.2}$$

$$I_{12}(\lambda) = I_1(\lambda) - I_2(\lambda) = \oint_{l_1\cup l_2}\left(\frac{\partial H}{\partial\lambda}\right)dt$$

are valid for α_i', α_i''. Here k_5 and k_6 are positive constants and the integrals in (3.2) are taken over the separatrices of the unperturbed problem. All the constants (δ_0 and k_i) can be chosen independent of Λ satisfying the inequality $\lambda_* < \gamma < \Lambda < k$ for arbitrary γ and k.

Theorem 1 shows that points starting from the interval d make a further turn in G_1, from interval d_2 are captured in G_2, from interval d_{12} are captured in G_{12}. The segments

$$[0, \alpha_3''],\ [\alpha_2', \alpha_2''],\ [\alpha_1', \alpha_1''] \tag{3.3}$$

"omitted" here have a common length $O(\delta^{3/2})$, while the whole interval $(0, \alpha_1'')$ has a length $O(\delta)$. The technique of proof of Theorem 1 is based on estimates of the integrals along undeterminate curves close to the separatrices, using the integrals in (3.2) along the separatrices of the unperturbed problem. Suppose that the point starts to move when $\lambda = \Lambda$ from the ray Cx, having $H = H(\Lambda) = O(\delta)$. For $\Lambda \leqslant \lambda \leqslant \Lambda_1$ this point describes a curve l_1' close to l_1 and either once again goes onto Cx or intersects l_1 and goes onto the semicircle $e = e_C$, $y > 0$. Here

$$H(\Lambda_1) = H(\Lambda) + \delta \int\limits_{l_1'} \left(\frac{\partial H}{\partial \lambda}\right) dt = H(\Lambda) - \delta I_1(\Lambda) + O(\delta^{3/2})$$

If $H(\Lambda) \in d$, then $H(\Lambda_1) > 0$ and, hence, for $\lambda = \Lambda_1$ the point lies in G_1, i.e. makes a complete turn. If $H(\Lambda) \in d_1$, then $H(\Lambda_1) < 0$, i.e. for $\lambda = \Lambda_1$ the point lies in G_{12}. In this case, for $\Lambda_1 \leqslant \Lambda \leqslant \Lambda_2$ the point describes a curve close to l_2 and $H(\Lambda_2) = H(\Lambda) - \delta [I_1(\Lambda) - I_2(\Lambda)] + O(\delta^{3/2})$. If $H(\Lambda) \in d_2$, then $H(\Lambda_2) > 0$ and the point falls into G_2 and remains there. If $H(\Lambda) \in d_{12}$, then $H(\Lambda_2) < 0$ and for $\lambda = \Lambda_2$ the point lies on the semicircle $e = e_C$, $y < 0$. The integrals in (3.2) are improper (motion along a separatrix requires infinite time), but converge. They can be computed.

Lemma.

$$I_1(\lambda) = (2\pi - \Theta) / 2, \quad I_2(\lambda) = \Theta / 2 \qquad (3.4)$$
$$\Theta = \arccos [(\lambda / 2x_C^2) - 2]$$

Here Θ is the angle formed by the tangents to l_1 at point C, $0 \leqslant \Theta < \pi$, $x_C = x_C(\lambda)$ is the largest root of the cubic equation (2.1). From (3.4) we see, in particular, that $I_1 > I_2$ and, consequently, the interval d_{12} is not empty for sufficiently small δ. For points with $H(\Lambda) \in d_{12}$, under motion in G_{12}, H receives an increment $\Delta H = -\delta [I_{12} + O(\delta^{1/2})] < 0$ at every turn as long as $H = O(\delta)$, $\Lambda_2 \leqslant \lambda \leqslant k$. Therefore, the point is immersed in G_{12}, leaves the separatrix and cannot go onto it once again. For proving the theorem it is essential that the point should not approach "too" close to C. Points not satisfying this condition belong to segments (3.3) and are excluded from consideration.

Note. The proof of Assertion 1 also is based on the property $I_1 > I_2$.

Let $(x(0), y(0)) \in G_{12}$ at the initial instant. Then by virtue of Assertion 2 the point $(x(t), y(t))$ intersects ray Cx for the last time for some $\lambda = \Lambda$.

Definition. We say that the point $(x(t), y(t))$ is captured in G_{12} if it lies in G_{12} for $\lambda = \Lambda + 1$. Otherwise the point is captured in G_2.

From Theorem 1 it follows that if δ is sufficiently small, then the points passing through interval d_{12} are captured in G_{12}, while those passing through d_2 are captured in G_2.

4. Probability of capture and its computation. From Theorem 1 it follows that a change in the value of H by a small quantity of the order of δ after the last turn can lead to a qualitative change in the character of the motion. Therefore, if the initial conditions are given to within ε $(\delta \ll \varepsilon \ll 1)$, it is not possible to indicate unambiguously in which of the regions G_2, G_{12} the evolution will take place after the intersection with the separatrix. In an asymptotic analysis of the problem $(\delta \to 0)$ it is appropriate to treat capture by some region or other as an event and to introduce its probability. Such an approach was applied in the capture problem for the oscillation of a pendulum perturbed by a small dissipative moment (*) [3] and in the problem of motion over an integral surface having a saddle point [4] (the latter problem is close to ours). The corresponding probabilities were computed in these papers.

The concept of random capture in the problem being examined was introduced in [1]. From the text of [1] we can realize, although this is not clearly stated, that the probability of capture in G_{12} is (in terms of the present paper) the ratio of H_{12} — the upper bound of the values of H on ray Cx, for which the point is captured in G_{12} — to H_1 — the upper bound of the values of H on ray Cx, for which the point cannot make a complete turn in G_1. The computation of these quantities reduces to a boundary-value problem. This problem was solved in [1] by a numerical integration of the equations of perturbed motion (1.2).

Theorem 1 yields analytic estimates for the quantities H_{12} and H_1

$$| H_{12} - \delta I_{12} | < 0.5 k_5 \delta^{3/2}, \quad | H_1 - \delta I_1 | < 0.5 k_3 \delta^{3/2}. \qquad (4.1)$$

If by refining the concept of capture probability in G_{12}, introduced in [1], we define the probability as

$$P = \lim_{\delta \to 0} H_{12} / H_1 \qquad (4.2)$$

then inequalities (4.1) yield

$$P = I_{12} / I_1 = (I_1 - I_2) / I_1 \qquad (4.3)$$

Here $P = P(\lambda)$ and formulas (4.3) and (3.4) allow us to compute P if we are given Λ, i.e. the value of parameter λ, for which the point goes into G_1 at the last turn.

Resonance in whose neighborhood the motion of a pair of satellites is described by Hamiltonian (1.1) has been considered in [1] and it was shown that capture of the representative point in G_{12} signifies capture of the satellites into resonance, while capture in G_2 signifies departure from resonance. The assumption exists that in former times the pair Enceladus-Dione of Saturn's satellites intersected the resonance being examined without being captured, i.e. its representative point on the phase plane approached a separatrix and departed into G_2. The probability of this event (capture in G_2) was obtained in [1] by numerical integration for hypothetical parameters of the approach to the separatrix: $p = 0.81$. For those same parameters formula (4.3) yields $p = 1 - P = 0.82$. The agreement is good if we take into consideration that in the problem being examined $\delta^{1/2} \sim 10^{-2}$.

5. Another definition of the probability. For what follows it is useful to define the probability in terms of phase volume. For a Hamiltonian system the phase volume — area is preserved. Suppose that a volume ΔV_1 occurs in G as λ varies on the interval $(\Lambda, \Lambda + \Delta\lambda)$. Let a fragment ΔV_{12} of this volume be captured in G_{12}. We define the probability of capture in G_{12} as

$$R(\Lambda) = \lim_{\Delta\lambda \to 0} \lim_{\delta \to 0} \Delta V_{12} / \Delta V_1 \qquad (5.1)$$

(Intuitively, here $\Delta\lambda$ is the imprecision in our knowledge of the instant of approach to the separatrix; $\Delta\lambda \gg \delta$ and, therefore, the outer limit is taken over $\Delta\lambda$.)

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*) A similar problem was considered in B. V. Chirikov's dissertation "Nonlinear Oscillations in Near-Conservative Systems", Novosibirsk, 1959.

Lemma. Definitions (5.1) and (4.2) are equivalent: $R(\Lambda) = P(\Lambda)$.

This can be proved by calculating the flow of the phase volume through the intervals d_1 and d_{12} and by estimating their closeness to ΔV_1 and ΔV_{12}, respectively. Another approach is the direct use of (5.1) for calculating the probability. For this we introduce $S_1(\lambda) = \text{mes}\, G$, $S_\nu(\lambda) = \text{mes}\, G_\nu$ $(\nu = 2, 12)$, where $\text{mes}\, G_\nu$ is area on the phase plane. Using incompressibility, we can show that

$$\Delta V_1 = S_1(\Lambda + \Delta\lambda) - S_1(\Lambda)$$

$$\lim_{\delta\to 0} \Delta V_{12} = S_{12}(\Lambda + \Delta\lambda) - S_{12}(\Lambda)$$

$$R = \lim_{\Delta\lambda\to 0} \frac{S_{12}(\Lambda + \Delta\lambda) - S_{12}(\Lambda)}{S_1(\Lambda + \Delta\lambda) - S_1(\Lambda)} = \frac{dS_{12}/d\lambda}{dS_1/d\lambda}$$

Using the formulas

$$dS_\nu / d\lambda = I_\nu, \qquad \nu = 1, 2, 12 \tag{5.2}$$

we obtain $R = I_{12} / I_1$, Q.E.D. Formulas (5.2) prove the following calculation (which we carry out for $\nu = 1$). Let $\Gamma_1 = \Gamma_1(\varphi, \lambda)$ be the equation of l_1. Then $H(\Gamma_1, \varphi, \lambda) = 0$, $\partial\Gamma_1 / \partial\lambda = -(\partial H / \partial\lambda) / (\partial H / \partial\Gamma)$

$$S_1 = \int_0^{2\pi} \Gamma_1 d\varphi, \quad \frac{dS_1}{d\lambda} = -\int_0^{2\pi} \left(\frac{\partial H/\partial\lambda}{\partial H/\partial\Gamma}\right) d\varphi = -\int_{2\pi}^{0} \left(\frac{\partial H/\partial\lambda}{\varphi}\right) dt = -\oint_{l_1} \left(\frac{\partial H}{\partial\lambda}\right) dt = I_1$$

6. Plan for an approximate consideration of the problem.

If the motion takes place at a distance from the separatrix, we are justified in applying the method of averaging along the trajectories of the unperturbed problem [6] for describing approximately the variation of the "slow" variable (the function H in the problem under analysis). In Sect. 7 we make assertions which allow us to use this fact in a somewhat modified form also for solutions intersecting a separatrix. We present below the plan for analyzing the problem to which this modification leads.

In each of the regions G_ν $(\nu = 1, 2, 12)$ we consider the equation (the "averaged equation")

$$\frac{dh}{d\lambda} = \frac{1}{T} \oint \frac{\partial H}{\partial\lambda} dt \tag{6.1}$$

where T is the period in the unperturbed problem and the integral is taken along the trajectories of the unperturbed problem $H(x, y, \lambda) = h \neq 0$ when $\lambda = \text{const}$. Actually, (6.1) defines three different equations depending upon the subdivision region selected. When $h = 0$ (i.e. on the region's boundaries, namely, the separatrices) we set $dh / d\lambda = 0$. Here the right-hand sides remain continuous (but not differentiable).

Let $(x(0), y(0)) \in G_1$; then $H(x(0), y(0), \lambda_*) = H_0 > 0$. We consider Eq. (6.1) in G_1 and its solution $h = h_1(\lambda)$. For some $\lambda = \Lambda$ we get $h_1(\Lambda) = 0$. A distinctive branching arises at this instant. Using Eq. (6.1) in region G_2 with initial condition $h = 0$ for $\lambda = \Lambda$, we obtain the solution $h = h_2(\lambda)$ immersed in G_2 ($h_2(\lambda)$ grows). Using (6.1) in G_{12} with the same initial condition, we obtain the solution $h = h_{12}(\lambda)$ immersed in G_{12} ($h_{12}(\lambda)$ decreases). A typical behavior of $h(\lambda)$ is shown in Fig. 3, a.

1°. If after intersection with the separatrix the evolution takes place in G_ν $(\nu = 2, 12)$, then the formula

$$h(\lambda) = \begin{cases} h_1(\lambda), & \lambda \leqslant \Lambda \\ h_\nu(\lambda), & \lambda > \Lambda \end{cases} \tag{6.2}$$

describes the variation of H with an accuracy of the order of $\delta \mid \ln \delta \mid$ for times of the order of $1 / \delta$.

2°. Motion in G_{12} is realized with probability $P(\Lambda) = I_{12}(\Lambda) / I_1(\Lambda)$, while in G_2, with probability $1 - P(\Lambda)$.

These assertions will be justified in Sects. 7 and 8. Thus, the plan reduces to the use of the averaged Eq. (6.1) in G_1 up to the separatrix, to the calculation of capture probability on the separatrix, and to the subsequent use of (6.1) in G_{12} and/or G_2.

This same plan can be set forth differently, by using an adiabatic invariant. In each of the regions G_ν ($\nu = 1, 2, 12$) we consider the function $J = J(h, \lambda)$, namely, the area contained inside the trajectory of the unperturbed problem with $H = h$ at the "slow time" instant λ. It can be verified [6] that this function is an integral of Eq. (6.1) considered in that same region. This integral is called an adiabatic invariant.

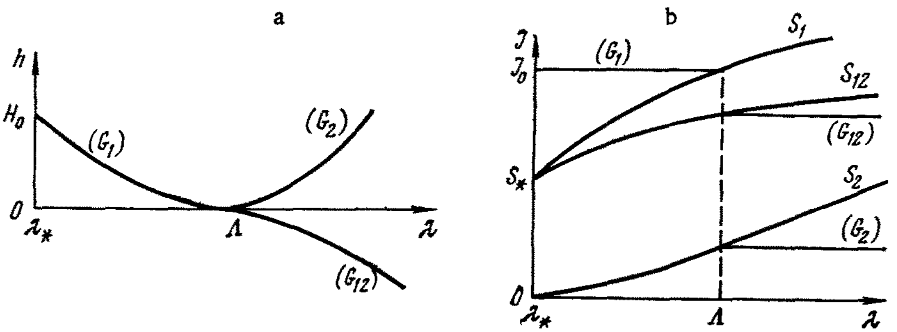

Fig. 3

Then:

The instant Λ is determined from the (transcendental) equation $J_0 = J(H_0, \lambda_*) = S_1(\Lambda)$; it has a single root since $J_0 > S_* = S_1(\lambda_*) = \pi\lambda_*$ and $dS_1 / d\lambda = I_1 > \pi / 2$.

For $\lambda \leqslant \Lambda$ the motion is determined approximately by the relation $J(H, \lambda) = J_0$.

For $\lambda > \Lambda$ the motion is described with probability $P(\Lambda)$ by the relation $J(H, \lambda) = S_{12}(\Lambda)$; since $dS_{12} / d\lambda = I_{12} > 0$, the trajectory is immersed in G_{12}. The motion is described with probability $1 - P(\Lambda)$ by the relation $J(H, \lambda) = S_2(\Lambda)$; since $dS_2 / d\lambda = I_2 > 0$, the trajectory is immersed in G_2.

This plan is presented graphically in Fig. 3, b.

For describing the motion with initial value $\lambda = \lambda_0 \neq \lambda_*$ we consider the area $W = W(f, \lambda)$ contained inside the trajectory with $F = f$. For $\lambda \leqslant \lambda_*$ this function is introduced in the whole plane, while for $\lambda > \lambda_*$, in each of the subdivision regions and $W(F, \lambda) = J(F - F_c, \lambda)$. Suppose that $F = F_0$ and $W = W_0 = W(F_0, \lambda_0)$ have been given for $\lambda = \lambda_0$. If $W_0 < S_*$, then an "automatic capture" is effected, the solution does not intersect the separatrix, and the averaging method in the usual form is applicable. We can use the formula $W(F, \lambda) = W_0$ for describing the motion. If $W_0 > S_*$, the previous plan is applicable with J replaced by W. We do not consider the exceptional case $W_0 = S_*$ in detail; the estimates on the accuracy are different here.

7. Substantiation of the averaging method procedure. Let us recall first of all what the averaging method in the usual formulation yields in the problem being examined. Let the motion start in G_1 and let $H(\lambda) = H(x(t), y(t), \lambda)$, $(t = (\lambda - \lambda_*) / \delta)$ be a solution of the exact problem (1.2), while $h_1(\lambda)$ is a solution of Eq. (6.1) in G_1 with like initial conditions: $H(\lambda_*) = h_1(\lambda_*) = H_0$.

Theorem [6]. For any $k_1 > 0$ there exist $\delta_0 > 0$ and $k_2 > 0$, such that $|H(\lambda) - h_1(\lambda)| < k_2\delta$ for $\delta < \delta_0$ as long as $h_1(\lambda) > k_1^{-1}$.

The restriction $h_1(\lambda) > k_1^{-1} > 0$ does not allow us to use the method close to the separatrix. It was found that in the problem being examined and, in general, in problems of similar type, the theorem is valid, permitting the use of the averaging method up to the separatrix.

Theorem 2. Positive constants δ_0, k_1, k_2, depending on H_0, exist such that

$$|H(\lambda) - h_1(\lambda)| < k_2\delta \tag{7.1}$$

for $\delta < \delta_0$ as long as $h_1(\lambda) > k_1\delta$.

Thus, the averaged equation can be used up to the δ-neighborhood of the separatrix. To prove this we consider the motion of a point in G_1 along the turns. Suppose that at the end of the n-th turn $H = H_n$, $\lambda = \lambda_n$, $J = J_n$, $t = t_n$. The following estimates are valid (k_i are positive constants) as long as the conditions

$$H > \delta, \qquad h_1(\lambda) > \delta \tag{7.2}$$

are fulfilled:

1°. $H_n - H_{n+1} > k_3^{-1}\delta$ is fulfilled for any n. Therefore,

$$H_{n-q} > H_n + qk_3^{-1}\delta, \quad n < k_3(H_0 - H_n) / \delta \tag{7.3}$$

2°. On the n-th turn

$$|H - H_n| < k_4\delta, \quad |\lambda - \lambda_n| < k_5\delta(1 + |\ln H_n|) \tag{7.4}$$

$$|J - J_n| < k_6\delta(1 + |\ln H_n|)$$

3°.
$$J_n - J_{n-1} = \int_{t_{n-1}}^{t_n} \frac{dJ}{dt} dt - \oint \frac{dJ}{dt} dt$$

where the last integral is taken along a trajectory of the unperturbed problem $H = H_n$, $\lambda = \lambda_n$; it is identically equal to zero since J is an integral of the averaged equation. Estimating the right-hand side of this relation with the aid of (7.4), we obtain

$$|J_n - J_{n-1}| < k_7\delta^2 H_n^{-1} \tag{7.5}$$

4°. Using (7.5), (7.3), (7.2), we obtain

$$|J_m - J_0| < \sum_{n=1}^{m} |J_n - J_{n-1}| < k_7\delta^2 \sum_{n=1}^{m} H_n^{-1} = \tag{7.6}$$

$$k_7\delta^2 \sum_{q=0}^{m-1} H_{m-q}^{-1} < k_7\delta^2 \sum_{q=0}^{m-1} (H_m + qk_3^{-1}\delta)^{-1} < k_7\delta^2 \times$$

$$\left[H_m^{-1} + \int_0^m (H_m + qk_3^{-1}\delta)\, dq\right] < k_7\delta\left[1 + k_3 \ln \frac{H_m + mk_3^{-1}\delta}{H_m}\right] <$$

$$k_7\delta\left[1+k_3\ln\frac{H_0}{H_m}\right]<k_8\delta(1+|\ln H_m|)$$

5°. From (7.4), (7.6), (7.2) it follows that on the m-th turn

$$|J-J_0|<|J-J_m|+|J_m-J_0|<(k_6+k_8)\delta(1+|\ln H_m|)<k_9\delta(1+|\ln H(\lambda)|)$$

6°. Since $J(h_1(\lambda),\lambda)=J_0$, then

$$|J(H(\lambda),\lambda)-J(h_1(\lambda),\lambda)|<k_9\delta(1+|\ln H(\lambda)|)$$

Using this inequality and (7.2), we obtain

$$|H(\lambda)-h_1(\lambda)|<k_2\delta \tag{7.7}$$

Conditions (7.2) cannot be violated as long as $h_1(\lambda)>(k_2+1)\delta=k_1\delta$, and we can guarantee the estimate (7.7) indicated. Analogous assertions are valid in regions G_{12} and G_2 For example, in G_{12} we have –

Theorem 2'. Let the constants $k>\Lambda'>\lambda_*$ be specified. There exist positive constants δ_0, k_{10}, k_{11} such that the estimate

$$|H(\lambda)-h_{12}'(\lambda)|<k_{10}\delta|\ln\delta| \tag{7.8}$$

is valid for $\delta<\delta_0$ as long as $\lambda<k$, where $H(\lambda)$ is the solution of the exact equations, while $h_{12}'(\lambda)$ is the solution of the averaged equation with like initial values: $h_{12}'(\Lambda')=H(\Lambda')<-k_{11}\delta$.

The proof is analogous to the preceding one, except that instead of (7.6) we obtain $|J_m-J_0|<k_{12}\delta(1+|\ln H(\Lambda')|)<k_{13}\delta|\ln\delta|$ and hence instead of (7.7) we arrive at (7.8).

Corollary. Let U be a closed subregion of G_1 for $\lambda=\lambda_*$, $k>\lambda_*$ be a constant. There exist positive constants δ_0, k_{10}, k_{11} such that when $\delta<\delta_0$, $\lambda<k$ the estimate

$$|H(\lambda)-h(\lambda)|<k_{14}\delta|\ln\delta| \tag{7.9}$$

($h(\lambda)$ is from (6.2)) is fulfilled for the points captured in G_ν ($\nu=2,12$) if $(x(0), y(0))\in U\setminus u$, where $u\subset U$ is some "exceptional" set: $\operatorname{mes} u<k_{15}\delta$.

To prove this we introduce $k_{16}>k_i+k_{11}$ uniformly with respect to $(x(0), y(0))\in U$ and Λ' from some interval. Suppose, for example, that the point is captured in G_{12}. For it we denote Λ_- and Λ_+ as the values of λ such that $H(\Lambda_-)=k_{16}\delta$, $H(\Lambda_+)=-k_{16}\delta$. If k_{16} is sufficiently large, then $\Lambda_-<\Lambda<\Lambda_+$ and

$$\Lambda_+-\Lambda_-<k_1\delta|\ln\delta| \tag{7.10}$$

for points passing "not too closely" to C (the latter guarantees the condition $(x(0), y(0))\notin u$, where $\operatorname{mes} u<k_{15}\delta$).

For $\lambda_*\leqslant\lambda\leqslant\Lambda_-$ estimate (7.9) follows from Theorem 2. $H(\lambda)=O(\delta)$ and $h(\lambda)=O(\delta)$ are fulfilled for $\Lambda_-\leqslant\lambda\leqslant\Lambda_+$, so that (7.9) is trivial. Theorem 2' is applicable for $\Lambda_+\leqslant\lambda\leqslant k$

$$|H(\lambda)-h_{12}'(\lambda)|<k_{10}\delta|\ln\delta|,\quad (h_{12}'(\Lambda_+)=-k_{16}\delta)$$

Using (7.10), we obtain

$$|h_{12}'(\lambda)-h(\lambda)|<k_{18}\delta|\ln\delta|$$

$$|H(\lambda)-h(\lambda)|<(k_{10}+k_{18})\delta|\ln\delta|=k_{14}\delta|\ln\delta|$$

Q.E.D. This corollary substantiates assertion 1° of Sect. 6.

8. Definition of the probability from the initial conditions. In (x, y, λ)-space we consider a point $M(x_0, y_0, \lambda_*)$ for which $H_0 = H(x_0, y_0, \lambda_*) > 0$. Let V be an ε-neighborhood of it. In V we pick out a subset V_{12} of points captured in G_{12}.

Definition. The expression

$$Q(M) = \lim_{\varepsilon\to 0} \lim_{\delta\to 0} \text{ mes } V_{12} / \text{mes} V \quad (8.1)$$

where mes V is the volume in (x, y, λ)-space, is called the probability of capture of point M in G_{12}. This definition is a formalization of the usual "physical" definition of the probability. It is borrowed from [5] with minor modifications.

Theorem 3. $Q(M) = P(\Lambda)$, where $P(\lambda)$ is from (4.3) and Λ is determined from the equation $S_1(\Lambda) = J(H_0, \lambda_*)$.

This theorem substantiates assertion 2° of Sect. 6. The proof is obtained by combining the results of Sects. 5 and 7.

9. Influence of small perturbations. Together with the system defined by Hamiltonian (1.3) we consider the proximate system of equations

$$x^{\cdot} = \partial F' / \partial y + \varkappa\delta g_1, \quad \dot{y} = -\partial F' / \partial x + \varkappa\delta g_2, \quad F' = F + \varkappa F_1 \quad (9.1)$$

where F_1, g_1, g_2 are smooth and bounded functions of x, y, λ; $\varkappa > 0$ is a constant. The probability $Q_\varkappa(M)$ of capture of a point M in region G_{12} is defined for system (9.1) in the same way as in Sect. 8.

Theorem 4. There exist $\varkappa_0 > 0$, $k_1 > 0$, such that when $\varkappa < \varkappa_0$ the probability $Q_\varkappa(M)$ exists and

$$| Q_\varkappa(M) - Q(M) | < k_1\varkappa \quad (9.2)$$

The proof is cumbersome; it is based on the fact that for system (9.1) we can argue as in Sects. 3, 7, 8 and obtain for $Q_\varkappa$ formulas of the type of Theorem 3. Estimates relative to these formulas lead to (9.2). By virtue of Theorem 4 the probability changes but little under the small perturbations which are always discarded when deriving equations of type (1.2) (see [1, 2]).

10. Pendulum problem. Let us now consider the pendulum problem (0.1). We assume $L = \delta\beta$, $\beta > 0$ is a constant, $\omega = \omega(\lambda)$, $\omega' = d\omega/d\lambda > 0$, $\lambda^{\cdot} = \delta$, $0 < \delta \ll 1$. We write the pendulum's equation of motion

$$\theta^{\cdot\cdot} + \omega^2 \sin\theta = -\delta\beta \quad (10.1)$$

It can be shown that the pendulum, originally moving in the direct rotation, is retarded and with probability

$$P = \begin{cases} 8\omega' / (4\omega' + \pi\beta), & \omega' < \pi\beta / 4 \\ 1, & \omega' \geqslant \pi\beta / 4 \end{cases} \quad (10.2)$$

is captured into oscillations. Here $\omega' = \omega'(\Lambda)$, where Λ is the instant of approach to the separatrix. Equation (10.1) was applied in [7] for the analysis of the evolution of the orbits of Saturn's resonant satellites. The capture probability value $P = 0.04$ was obtained in [1], using such a model, for the pair of satellites Mimas-Tethys by a numerical integration of (10.1). For those same problem parameters formula (10.2) yields

$P = 0.044.$

A plan, analogous to that presented in Sect. 6, is applicable for describing the evolution. We introduce the unperturbed problem's Hamiltonian $H = (\theta^{\cdot 2}/2) - 2\,\omega^2 \cos^2 (\theta / 2)$; $H > 0$ for rotations and $H < 0$ for oscillations. For $H \geqslant 0$ we introduce

$$J(H, \lambda) = \int_0^{2\pi} |\theta^{\cdot}|\, d\theta = \frac{8\omega}{z} E(z), \quad z = \frac{2\omega}{\sqrt{2H + 4\omega^2}}$$

where $E(z)$ is the complete elliptic integral of the second kind with modulus z. By virtue of the averaged equation, $dJ / d\lambda = -2\pi\beta$ in the region of direct rotation, and for $\lambda_0 \leqslant \lambda \leqslant \Lambda$ the evolution is described by the formula $J(H, \lambda) = J(H_0, \lambda_0) - 2\pi\beta(\lambda - \lambda_0)$. The instant Λ is determined from the condition $J(0, \Lambda) = S(\Lambda) / 2$, where $S(\lambda) = 16\,\omega(\lambda)$ is the area of the oscillatory region on a segment of length 2π. Therefore, $8\,\omega(\Lambda) + 2\pi\beta(\Lambda - \lambda_0) = J(H_0, \lambda_0)$.

Shift into the reverse rotation takes place with probability $1 - P(\Lambda)$. If $1 - P \times (\Lambda) > 0$, this evolution is determined by the formula $J(H, \lambda) = 2\pi\beta(\lambda - \Lambda) + 8\,\omega(\Lambda)$. Capture in the oscillatory region takes place with probability $P(\Lambda)$. Here we introduce

$$J(H, \lambda) = \oint |\theta^{\cdot}|\, d\theta = 16\,\omega\,[E(z) - (1 - z^2)\,K(z)]$$

where

$$z = \sqrt{1 + H / 2\omega^2} \quad (H < 0)$$

$K(z)$ and $E(z)$ are the complete elliptic integrals of the first and second kind, respectively, with modulus z. The formula $J(H, \lambda) = S(\Lambda) = 16\,\omega(\Lambda)$ describes the motion in the oscillatory region.

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